

①

Kuga-Satake construction
and
moduli spaces of
lattice polarised K3 surfaces

Flora Poon
NCTS, Taiwan

Goal

Give explicit map

$$\mathcal{K}_P \xrightarrow{F} \mathcal{A}_{8,D,H}$$

moduli space of
P-polarised
K3 surfaces,
P: rk 14 lattice

moduli space of
abelian 8-folds
with polarisation type D
 $\text{End}_{\mathbb{Q}}(A) \supset F$, $F_{\mathbb{R}} \simeq H$

(**): det. of quad. form
on $P \subset \text{sgm } \mathbb{Q}^*$

Settings

③

[1] Any K3 X is an algebraic K3/ $\mathbb{C}$
i.e. compact, simply connected, trivial canonical bundle, irregularity zero

- P : lattice of rk r , signature $(1, r-1)$

$$\text{s.t. } P \xrightarrow{\text{primitive}} \underbrace{\Delta_{K3}}_{\substack{S \\ H^2(X, \mathbb{Z})}} \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

(rough) Def X has a P -polarisation if

$$\exists \quad P \xrightarrow{\text{primitive}} \text{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \cap H^{1,1}$$

s.t. $\text{Im}(P) \ni$ ample line bundle

- $H^2(X, \mathbb{Z})$ has a wt 2 Hodge structure (HS) ④

i.e. $H^2(X, \mathbb{C}) \simeq H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$, $H^{0,2}(X) = \overline{H^{2,0}(X)}$

$\downarrow$
 $\text{dim} \quad 1 \quad \vee \quad 20 \quad 1$
 $\quad \quad P$

determined by $H^{2,0}(X) \subset T_{\mathbb{C}}$

$$T := P_{\Delta_{K3}}^{\perp}$$

"transcendental lattice"

defined
up to
isometry $\rightarrow$

Note V has HS:

$$V_{\mathbb{C}} \simeq V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$$

dim $\quad 1 \quad 4 \quad 1$

then V has HS
"of K3 type"

* Assume all $P \xrightarrow{\text{primitive}} \text{Pic}(X)$ lie in same orbit of $O(\Delta_{K3})$

5

- The Global Torelli Theorem is true for $\mathcal{K}_P$

i.e. $\mathcal{K}_P \simeq \frac{\{ \text{wt } 2 \text{ HS on } H^2(X, \mathbb{Q}) \}}{\text{Hodge isometries}} \xrightarrow{\text{Period domain}} \mathcal{D}_{\mathcal{K}_P}$

$\simeq \frac{\Gamma(P) \backslash SO^+(2, 20-r)}{SO(2) \times SO(20-r)}$

Because: $SO^+(2, 20-r) \overset{\text{transitive}}{\curvearrowright} \mathcal{D}_{\mathcal{K}_P}$

stabiliser subgroup: $SO(2) \times SO(20-r)$

grp of Hodge isom.: $\Gamma(P) \underset{\text{arithmetic}}{\leq} SO^+(2, 20-r)$

⑥

2 abelian variety $A = (\Pi, J, L)$

$$\Pi = \frac{\Delta_{\mathbb{R}}}{\Delta}, \quad \Delta = \text{lattice} \subset \mathbb{R}^{2g}$$

$J =$ Complex structure of $\Delta_{\mathbb{R}}$
 $\Downarrow$

wt 1 HS on $H^1(A, \mathbb{Z})$

$L =$ polarisation = ample line bundle
 $\nleftrightarrow$ derive

matrix D "polarisation type"

(rough)

7

Def

An endomorphism structure
of an abelian variety A is

$$\hookrightarrow F \hookrightarrow \text{End}_{\mathbb{Q}}(A) \hookrightarrow \rho$$

s.t. anti-invol. ι extends to the Rosati invol. ρ

$$\iota(xy) = \iota(y)\iota(x)$$

anti invol
induced by pol. ϕ_D
 $\rho(f) = \phi_D^{-1} \circ f \circ \phi_D$

(2)

Goal

Give explicit map

$$\begin{array}{ccc}
 \mathcal{K}_P & \xrightarrow{F} & \mathcal{A}_{8,D,H} \\
 \text{moduli space of} & & \text{moduli space of} \\
 P\text{-polarised} & & \text{abelian 8-folds} \\
 K3 \text{ surfaces,} & & \text{with polarisation type } D \\
 P: \text{rk } 14 \text{ lattice} & & \boxed{\text{End}_{\mathbb{Q}}(A) \supset F, \text{ } F_{\mathbb{R}}^{\vee} = H} \\
 & & \text{ } \quad \quad \quad \vee: \text{conjugation}
 \end{array}$$

"totally definite quaternion mult."

⑧

- Torelli Thm is also true for $A_{2m, D, H}$ ^{ab. (2m)-fold}

$$A_{2m, D, H} \cong \cong \{ \text{wt 1 HS on } H^1(A, \mathbb{Z}) \} \leftarrow \begin{matrix} \text{Period} \\ \text{domain} \\ D_{A_{2m, D, H}} \end{matrix}$$

$$\cong \Gamma_{(2m, D, H)} \xrightarrow{SO^*(2m)} U(m)$$

$$\begin{matrix} \wedge \text{arith.} \\ SO^*(2m) \end{matrix}$$

$$Sp(2m, \mathbb{C}) \cap \text{Aut}_H(\mathbb{R}^{2m}) \subset M_{2m}(\mathbb{R})$$

aut. of $\mathbb{R}^{2m}$ as H -mod.

Motivation

9

Fact

For $r = \dim(P) \geq 14$

$$\begin{array}{ccc} \nearrow & D_{K_P} & \xrightarrow{\text{isom}} D_M \\ \text{period domain of } K_P & & \text{period domain of mod. sp. } M \\ & & \text{of some diff. (pol.) geom. objects} \end{array}$$



r	M
20	(supersingular) points
19	modular curves
18	Hilbert modular surfaces
17	modular varieties of polarised abelian surfaces with level structure
16	modular varieties of deformation of generalised Kummer varieties
15	modular varieties of deformation of hyperkähler manifolds of type OG6
14	modular varieties of abelian 8-folds with totally definite quaternion multiplication

$= X_{8,D,H}$

$$\underline{r=14}$$

10

$$\overset{\substack{SO^*(2,6) \\ / \\ SO(2) \times SO(6) \\ 12}}{\mathcal{D}_{\mathcal{K}_P}} \xrightarrow{\sim F} \mathcal{D}_{\mathcal{A}_{8,L,H}} \overset{\substack{SO^*(8) \\ / \\ U(4)}}{\simeq}$$

$$\downarrow \Gamma(P)$$

$$\downarrow \Gamma^*(P)$$

$$\boxed{\mathcal{K}_P \xrightarrow{\#} \mathcal{A}_{8,L,H}}$$

dim 6

dim 6

$\sim$
F equivariant

More Background

11

$$R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$$

$(V, g): R\text{-mod}/v.\text{space} + \text{quadratic form}$

Def

Clifford algebra over V

$$\text{Cl}(V) := \frac{T(V)}{I(V)} \begin{array}{l} \leftarrow \text{tensor alg.} \\ \leftarrow \text{relation on multiplication} \\ v \otimes v = g(v) \quad \forall v \in V \end{array}$$

Notation

$$\text{If } V \simeq \mathbb{R}(n_+, n_-)$$

$$\text{Then write } \text{Cl}(n_+, n_-) := \text{Cl}(V)$$

Properties of Clifford algebra

12

- $Cl(V) \simeq \underset{\text{even}}{Cl^+(V)} \oplus \underset{\text{odd}}{Cl^-(V)} \quad (\mathbb{Z}_2\text{-grading})$
- $Cl(V) \underset{\text{graded algebra}}{\simeq} \text{product of matrix algebras}$

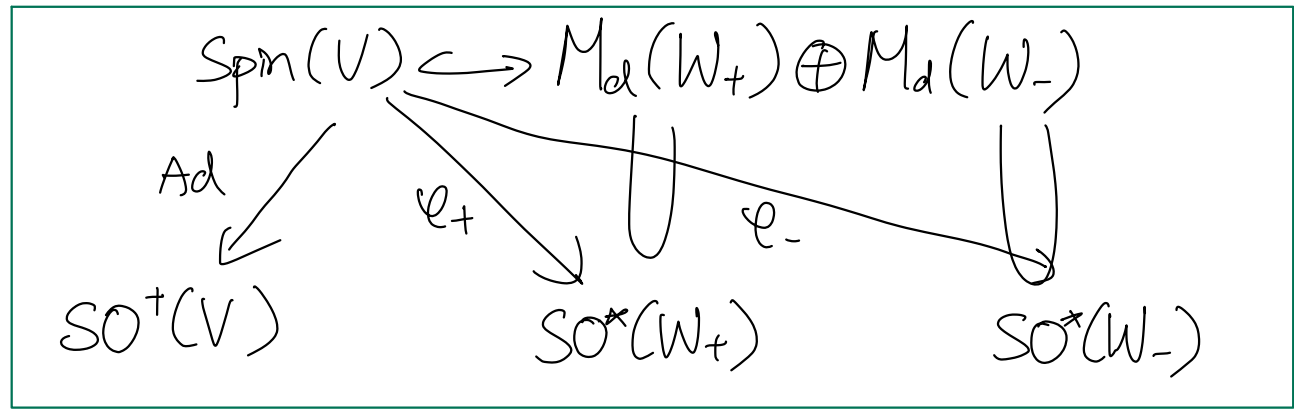
EX

$$Cl^+(2,6) \simeq Cl(2,5) \simeq M_4(\mathbb{H}) \times M_4(\mathbb{H})$$

13

• $\text{Spin}(V) < \text{Cl}^*(V) \cap \text{Cl}^+(V)$
 spin grp $\nearrow$ grp of units

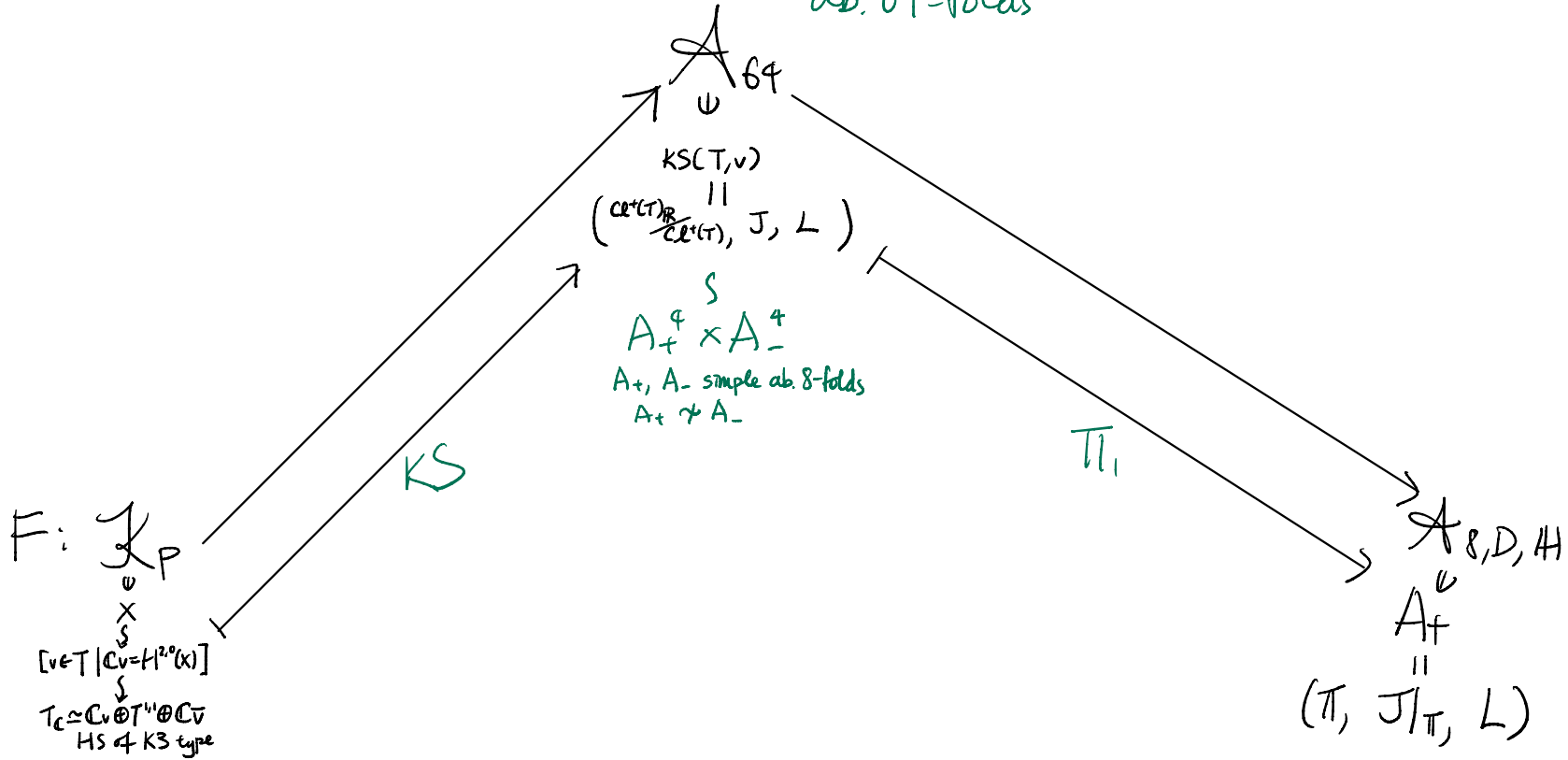
When $V \simeq \mathbb{R}^n$, $\text{sig} = (n_+, n_-)$ where $n_+ - n_- \equiv 0 \pmod 4$
 $\text{Spin}^+(V)$ has 3 inequivalent representations
 identity component $\nearrow$



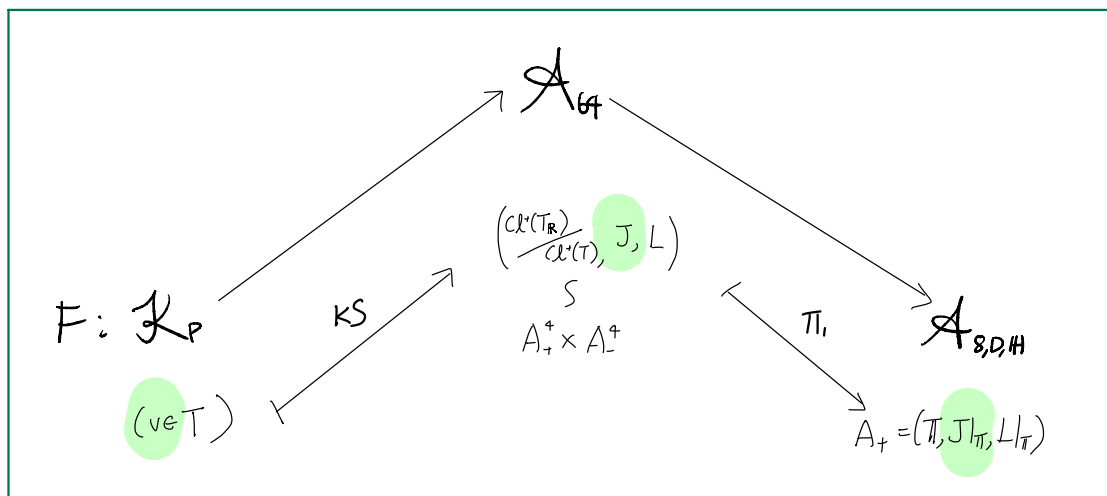
Construct F

14

mod of
ab. 64-folds



15



Rmk

- $F: [v] \mapsto [J] \mapsto [J|_{\pi}]$
- F descends from $\tilde{F}: D_{K_P} \longrightarrow D_{A_{8,D,H}}$
 $v \mapsto J \mapsto J|_{\pi}$

F descends from $\tilde{F} : D_{\mathbb{R}_p} \longrightarrow D_{\mathbb{A}_{8,D,H}}$
 $V \mapsto J \mapsto J|_{\Pi_i}$

16

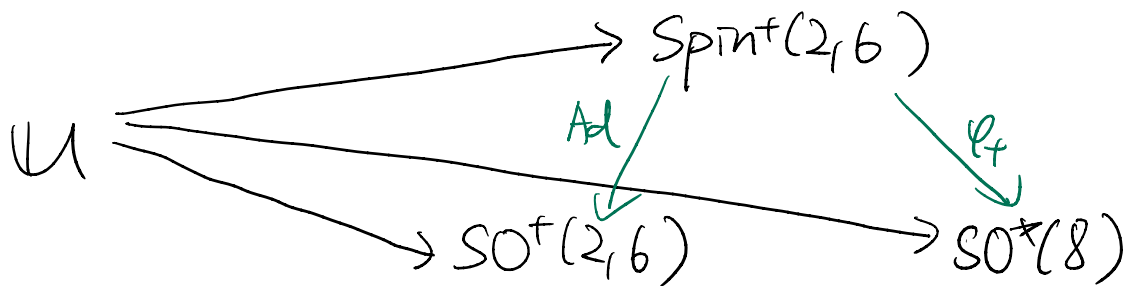
Explanation

V has HS

$\Updownarrow$

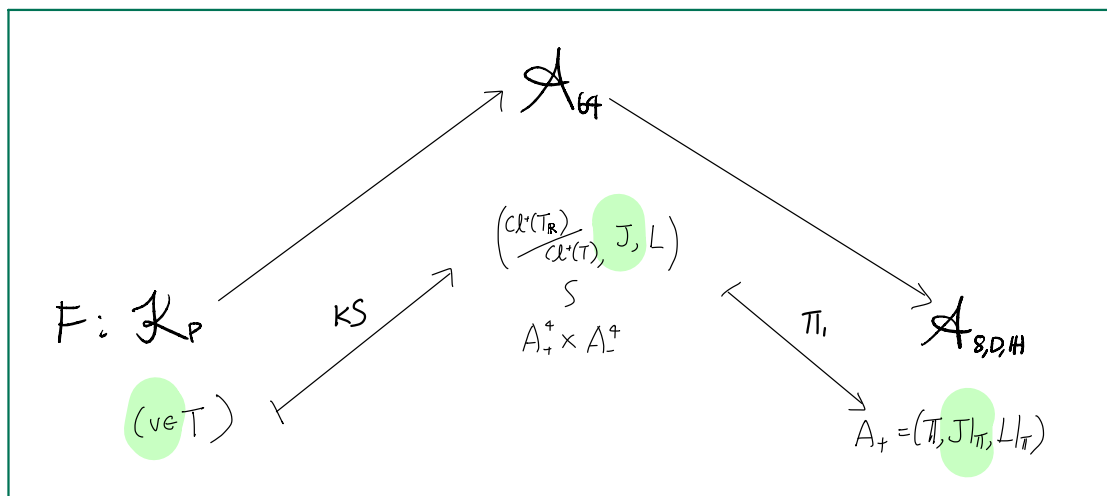
$\exists$ grp homo $U \longrightarrow \text{Hdg}(U) \longrightarrow GL(V)$
 ii
 $\{x \in \mathbb{C}^* \mid N_m(x) = 1\}$

$\Rightarrow$



equivariant!

15

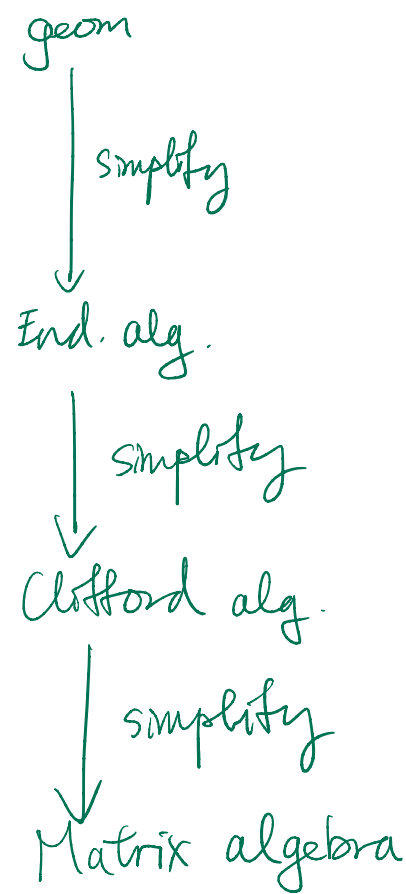


Rmk

- $F: [v] \mapsto [J] \mapsto [J|_{\pi}]$
- F descends from $\tilde{F}: D_{\mathcal{K}_P} \longrightarrow D_{A_{8,D,H}}$
 $v \mapsto J \mapsto J|_{\pi}$
- F is a holomorphic local diffeomorphism
 $\rightarrow$ extends to Satake-Baily-Borel comp.

What is left:

17



$$KSCT, v) \sim \underbrace{A_1 \times \dots \times A_4}_{\sim A_+} \times \underbrace{A_5 \times \dots \times A_8}_{\sim A_-}$$

$$\frac{Cl^+(T) \otimes \mathbb{R}}{Cl^+(T)}$$

$$\simeq A_i = \text{Im}(d_i \varepsilon_i)$$

$$\frac{(\Delta) \otimes \mathbb{R}}{\Delta_1}$$

$$\text{End}(KS(T)) \underset{\text{Fact}}{\simeq} Cl^+(T)$$

$$\Delta_1 = d_i \varepsilon_i \cdot Cl^+(T)$$

$$\uparrow$$

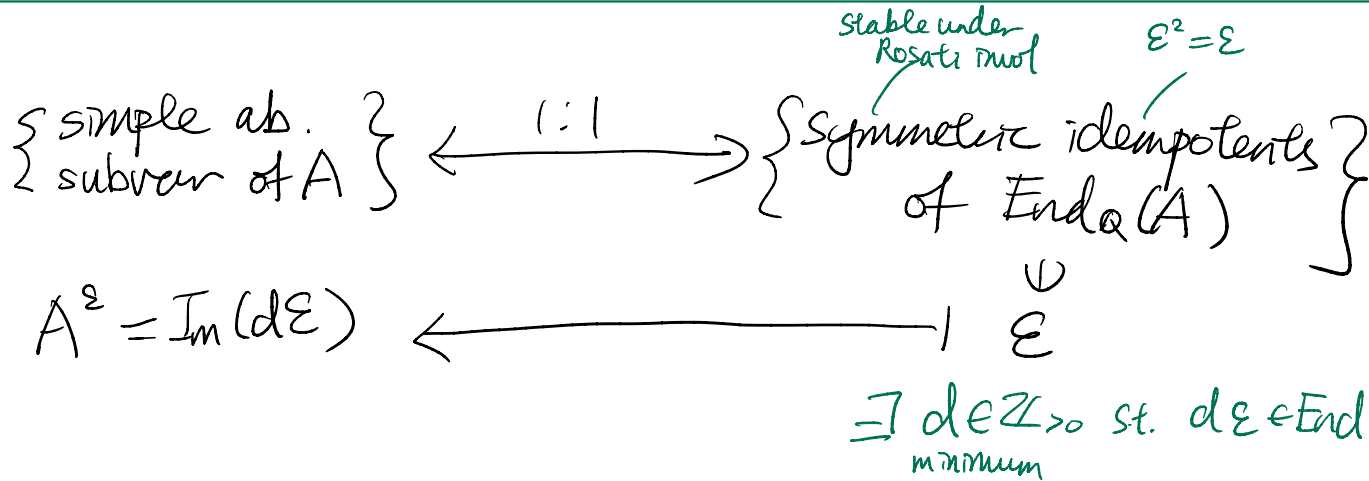
$$Cl^+(T)$$

$$\text{fix } Cl_{\mathbb{Q}}^+(T_{\mathbb{Q}}) \underset{(*)}{\simeq} M_4(F) \times M_4(F), \quad F_{\mathbb{R}} = \mathbb{H}$$

Facts on simple decomp. of ab. variety A

18

Fact 1



Facts on simple decomp. of ab. variety A

19

Fact 2 (Poincaré's complete reducibility thm)

$$A \sim \prod_i A_i^{d_i} \quad (A_i \text{ simple, } A_i \not\sim A_j \text{ if } i \neq j)$$

$$\Leftrightarrow \text{End}_{\mathbb{Q}}(A) \simeq \prod_i M_{d_i}(\text{End}_{\mathbb{Q}}(A_i))$$

$$(\ast\ast) \quad \text{Cl}_{\mathbb{Q}}^+(T) \simeq M_q(F) \times M_q(F), \quad F_{\mathbb{R}} \simeq \mathbb{H}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ d, \varepsilon_i & \longmapsto & \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \times \text{Id}_q \end{array} \quad \text{symmetric idempotent}$$

How to fix $Cl_{\mathbb{Q}}^{\epsilon}(T) \simeq M_4(H\mathbb{Q}) \times M_4(H\mathbb{Q})$? 20

Universal property of Clifford algebra

$$\varphi: V \xrightarrow{\text{mod-homo}} \mathcal{A} \text{ s.t. } \varphi(v)\varphi(v) = q(v) \cdot 1_{\mathcal{A}}$$

then φ extends ! to $Cl(V) \xrightarrow{\text{alg. homo}} \mathcal{A}$

Gluing of Clifford algebras graded tensor product

$$V \oplus V' \longrightarrow Cl(V) \overset{\wedge}{\otimes} Cl(V')$$

$$(v, v') \longmapsto v \otimes 1 + 1 \otimes v'$$

$$\text{extends to } Cl(V \oplus V') \xrightarrow[\cong]{\text{iso}} Cl(V) \overset{\wedge}{\otimes} Cl(V')$$

e.g. $T = U \oplus U(2) \oplus D_4(-1)$

enough to fix $Cl_{\mathbb{Q}}(U(n)) \simeq M_2(\mathbb{Q})$
 $Cl_{\mathbb{Q}}(D_4(-1)) \simeq M_2(H\mathbb{Q})$
 and glue.

Specialisation of F

(21)

Consider $\mathcal{L}_{P'} \subset \mathcal{K}_P$

$$\left\{ \begin{array}{l} X' \in \mathcal{K}_P : X' = P'\text{-polarised, } P' \text{ rk } 18, \\ P' \hookrightarrow P \hookrightarrow \Lambda_{k3} \text{ primitive} \\ T' := (P')^\perp_{\Lambda_{k3}} \subseteq U \oplus S \end{array} \right\}$$

$\Downarrow$ [Morrison]

X' has Shioda-Inose structure ass. to ab. surf. A'

$$\begin{array}{l} \sigma \odot X' \xrightarrow{\pi} \text{Kum}(A') =: Y \\ \text{i.e. } \begin{array}{l} \text{symplectic} \\ \text{mod} \end{array} \searrow \begin{array}{l} X' \\ \sigma \end{array} \swarrow \begin{array}{l} \text{min.} \\ \text{resol.} \end{array} \end{array} \quad \text{s.t. } \pi \text{ induces Hodge isom} \\ (\pi_x)_* : T_x(2) \rightarrow T_Y$$

$$\underline{\text{Thm}} \quad X' \in \mathcal{L}_{P'} \Rightarrow F(X') \sim (A')^4$$

22

$$\underline{\text{Partial proof}} : KS(X') \sim (A')^{2^5}$$

Fact 1 [Morrison]

$$T' \subset T \text{ HS of K3 type, } d := \dim\left(\frac{T_{\mathbb{Q}}}{(T')_{\mathbb{Q}}}\right)$$

then $KS(T) \sim (KS(T'))^{2^d}$

$$H^2(X, \mathbb{Z}) \cong \Delta_{K3}$$

rk 22

$$T := P_{\Delta_{K3}}^{\perp}$$

rk 8

$$T' := (P')_{\Delta_{K3}}^{\perp}$$

rk 4

$$(1) : KS(\Delta_{K3}) \sim KS(T')^{2^{18}}$$

$$(2) : KS(T) \sim KS(T')^{2^4}$$

Partial proof: $KS(X') \sim (A')^{2^5}$

23

Pf Fact 2 [Morrison] —
 X' has Shioda-Inose structure ass. to A'
 then $KS(H^2(X, \mathbb{Z})) \sim (A')^{2^{19}}$

$$(1): KS(\Delta_{k3}) \sim KS(T')^{2^{18}} \Rightarrow KS(T') = (A')^2$$

$$(2): KS(T) \sim KS(T')^{2^4} \Rightarrow KS(T) = (A')^{2^5}$$